THE LEAST AREA PROPERTY OF THE MINIMAL SURFACE DETERMINED BY AN ARBITRARY JORDAN CONTOUR

BY JESSE DOUGLAS

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Communicated March 3, 1931

1. In a recent paper[1] the writer gave the first solution of the problem of Plateau for an arbitrary Jordan contour, proving that if Γ is any Jordan curve in n-dimensional euclidean space there always exists a minimal surface M

$$x_i = RF_i(w), \qquad (1.1)$$
$$\sum_{i=1}^{n} F_i'^2(w) = 0, \; |w| < 1,$$

bounded by Γ.

If it is possible to span in the contour Γ at least one surface of the topological type of a circular disc having finite area, then the minimal surface M realizes the solution of the least area problem for the contour Γ: the area of M is finite and $\leqq$ the area of any continuous surface bounded by Γ.

An example was given[2] of a Jordan curve within which it is impossible to span any surface whatever of finite area, i.e., the area of every surface bounded by Γ including that of M is $+\infty$. It must not be thought, however, that the least area property of M becomes vacuous in such a case. On the contrary, the purpose of the present note is to show that there is a good sense in which this property continues to hold; this is done in the form of the following theorem.

THEOREM: *Every Jordan curve Γ in euclidean space of any number n of dimensions is the boundary of a surface M of the topological type of a circular disc having the following properties:*

(a) *M is a minimal surface, in the sense of the equations* (1.1).

(b) *If Γ' denote any continuous closed curve on M and lying altogether in its interior—the image by* (1.1) *of any Jordan curve C' whose points are all interior to the unit circle—then Γ' intercepts upon M a region whose inner area is finite and $\leqq$ the inner area of any other continuous surface bounded by Γ'.*

(c) *If the entire area of M is finite, this is $\leqq$ any other area bounded by Γ. If the area of M is $+\infty$, then the area of every continuous surface bounded by Γ is $+\infty$. The same remarks apply to the area bounded by any continuous closed curve upon M having points in common with Γ.*

It thus appears that in the case where M has infinite area the infinite part of the area lies, so to say, altogether on the edge of M.

2. Fundamental in the cited paper was the functional

$$A(g) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi}\frac{\sum_{i=1}^{n}[g_i(\theta) - g_i(\varphi)]^2}{4\sin^2\frac{\theta-\varphi}{2}}\,d\theta\,d\varphi,$$

where the range of the argument g consists of all parametric representations, $x_i = g_i(\theta)$, of the given contour Γ. With respect to this functional all contours divide themselves into two classes: those for which there exists a representation g such that $A(g)$ is finite, and those for which $A(g)$ is identically $+\infty$. Since it was shown that the minimum value of $A(g)$ is equal to the least area bounded by the given contour, we designated these two types as finite-area-spanning and non-finite-area-spanning, respectively. It was shown that any contour of the first type was the boundary of a minimal surface M having the least area bounded by that contour. The minimal surface M bounded by a non-finite-area-spanning contour was then obtained by considering Γ as a limit of polygons.

3. In beginning the proof of the theorem of this paper, we remark that if Γ is any finite-area-spanning contour and Γ_ρ is the analytic curve upon M corresponding by the equations (1.1) to the circle C_ρ of radius $\rho < 1$ about the origin as center, then Γ_ρ must intercept on M a smaller area than on any other continuous surface which it bounds. For if there is a surface of smaller area bounded by Γ_ρ, then by replacing the portion of M bounded by Γ_ρ by this smaller area we would have a surface of smaller area than M bounded by Γ.

4. Next, let Γ be a non-finite-area-spanning contour, represented as the limit of a sequence of polygons $\Gamma^{(m)}$. We saw in the cited paper that these polygons could be selected so that their representations minimizing $A(g)$,

$$x_i = g_i^{(m)*}(\theta), \tag{4.1}$$

tended to a representation of Γ,

$$x_i = g_i^*(\theta), \tag{4.2}$$

which when used in Poisson's integral,

$$x_i = RF_i(w),\ F_i(w) = \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\theta}+w}{e^{i\theta}-w}\,g_i^*(\theta)d\theta, \tag{4.3}$$

gave a minimal surface M bounded by Γ. This surface is the limit of the minimal surface $M^{(m)}$ bounded by $\Gamma^{(m)}$, defined by

$$x_i = RF_i^{(m)}(w),\ F_i^{(m)}(w) = \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{i\theta}+w}{e^{i\theta}-w}\,g_i^{(m)*}(\theta)d\theta. \tag{4.3$_m$}$$

M is the surface of which we assert the properties stated in our main theorem.

Let C_ρ be any circle concentric with and smaller than the unit circle and not passing through any singluar point of M; this means that at every point of C_ρ

$$\sum_{i=1}^{n} | F_i'(w) |^2 > 0. \tag{4.4}$$

Let Γ_ρ, $\Gamma_\rho^{(m)}$ be the analytic curves corresponding to C_ρ on M, $M^{(m)}$, respectively; these curves appear in definite topological representation upon C_ρ and therefore in definite point-to-point correspondence with one another. The parametric equations of Γ_ρ, $\Gamma_\rho^{(m)}$ are, respectively,

$$x_i = RF_i(\rho e^{i\theta}) = G_i(\theta), \tag{4.5}$$

$$x_i = RF_i^{(m)}(\rho e^{i\theta}) = G_i^{(m)}(\theta). \tag{4.5$_m$}$$

We have

$$A(G) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi} \frac{\sum_{i=1}^{n} [G_i(\theta) - G_i(\varphi)]^2}{4\rho^2 \sin^2 \dfrac{\theta - \varphi}{2}} . \rho d\theta . \rho d\varphi, \tag{4.6}$$

$$A(G^{(m)}) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi} \frac{\sum_{i=1}^{n} [G_i^{(m)}(\theta) - G_i^{(m)}(\varphi)]^2}{4\rho^2 \sin^2 \dfrac{\theta - \varphi}{2}} . \rho d\theta . \rho d\varphi. \tag{4.6$_m$}$$

(If we wish, we may consider the differentials of integration as $\rho d\theta$ and $\rho d\varphi$, elements of arc on C_ρ.) The integrands take for $\theta = \varphi$ the indeterminate form $\frac{0}{0}$; but it is easy to see that they have, respectively, the limiting values

$$\frac{1}{2}\sum_{i=1}^{n} | F_i'(w) |^2, \quad \frac{1}{2}\sum_{i=1}^{n} | F_i^{(m)\prime}(w) |^2, \tag{4.7}$$

for $\theta = \varphi (w = \rho e^{i\theta})$. One has only to interpret the integrand in (4.6) ((4.6$_m$)) as the square of the chord of $\Gamma_\rho(\Gamma_\rho^{(m)})$ divided by the square of the corresponding chord of C_ρ. The limit in question is evidently the square of the modulus of the conformal transformation from the circular disc to the minimal surface, and that is (4.7).

Let I denote the integrand of (4.6); with the definition (4.7) for $\theta = \varphi$, I is finite and continuous, therefore bounded, on the entire torus (θ, φ). It follows that $A(G)$, equal to *the area of* M_ρ, *the region of* M *bounded by* Γ_ρ, *is finite.*[3] (From the appraisal (4.10) below, which applies to I as well as I_m, it may be seen that the area of M_ρ is of the order of $(1 - \rho)^{-4}$.)

Let 2λ be the minimum value of $\frac{1}{2}\sum_{i=1}^{n} | F_i'(w) |^2$ on C_ρ; by (4.4), $2\lambda > 0$. By continuity, I stays greater than λ for values of (θ, φ) near enough to the diagonal $\theta = \varphi$ of the torus (θ, φ):

$$I > \lambda \text{ for } |\theta - \varphi| < \alpha. \tag{4.8}$$

We next prove that the integrand I_m of (4.6_m) stays uniformly bounded in θ, φ and m. Evidently, from (4.5_m) and (4.6_m),

$$I_m \leqq \frac{\sum_{i=1}^{n} | F_i^{(m)}(\rho e^{i\theta}) - F_i^{(m)}(\rho e^{i\varphi}) |^2}{4\rho^2 \sin^2 \frac{\theta - \varphi}{2}}. \tag{4.9}$$

From (4.4),

$$F_i^{(m)}(\rho e^{i\theta}) - F_i^{(m)}(\rho e^{i\varphi}) = \frac{1}{\pi}\int_0^{2\pi} \frac{e^{i\omega}(\rho e^{i\theta} - \rho e^{i\varphi})}{(e^{i\omega} - \rho e^{i\theta})(e^{i\omega} - \rho e^{i\varphi})} g_i^{(m)*}(\omega) d\omega,$$

whence

$$\frac{| F_i^{(m)}(\rho e^{i\theta}) - F_i^{(m)}(\rho e^{i\varphi}) |}{2\rho \sin \frac{\theta - \varphi}{2}} = \left| \frac{1}{\pi}\int_0^{2\pi} \frac{e^{i\omega}}{(e^{i\omega} - \rho e^{i\theta})(e^{i\omega} - \rho e^{i\varphi})} g_i^{(m)*}(\omega) d\omega \right|$$

since

$$| \rho e^{i\theta} - \rho e^{i\varphi} | = 2\rho \sin \frac{\theta - \varphi}{2}.$$

Obviously all the contours $\Gamma^{(m)}$ can be comprised in a cube of edge $2R$ with center at the origin; therefore

$$\frac{| F_i^{(m)}(\rho e^{i\theta}) - F_i^{(m)}(\rho e^{i\varphi}) |}{2\rho \sin \frac{\theta - \varphi}{2}} \leqq \frac{2R}{(1 - \rho)^2},$$

and consequently, by (4.9),

$$I_m \leqq \frac{4nR^2}{(1 - \rho)^4}. \tag{4.10}$$

Since I_m thus stays uniformly bounded as it approaches to I, it follows that

$$\lim_{m \to \infty} A(G^{(m)}) = A(G). \tag{4.11}$$

By (4.8) and (4.10) we have that

$$\frac{I_m}{I} < B \text{ for } 0 < |\theta - \varphi| < \alpha, \tag{4.12}$$

B and α being independent of m.

The above is preliminary to the proof of the assertion that $A(G)$ *is the least value of the A-functional for the contour* Γ_ρ. Suppose, on the contrary, that G^* is a parametric representation of Γ_ρ such that

$$A(G^*) < A(G). \tag{4.13}$$

G^* is derivable from G by a parameter transformation on Γ_ρ; let $G^{(m)*}$ be the representation of $\Gamma_\rho^{(m)}$ derived from $G^{(m)}$ by the same parameter transformation. The quotient $\frac{I_m}{I}$ in (4.12) is evidently the ratio of the squares of corresponding chords of $\Gamma_\rho^{(m)}$ and Γ_ρ, and corresponding chords remain such after the same parameter transformation is effected on both contours. It follows that, in an obvious notation,

$$\frac{I_m^*}{I^*} = \frac{I_m}{I} < B \text{ for } 0 < |\theta - \varphi| < \alpha, \tag{4.14}$$

B and α being the same as in (4.12).

Now $A(G)$ being finite, so also is $A(G^*)$, and therefore

$$A_\epsilon(G^*) = \frac{1}{4\pi}\int\int_{T_\epsilon} I^*.\rho d\theta.\rho d\varphi \tag{4.15}$$

—where T_ϵ is the domain (θ, φ) where $|\theta - \varphi| \geqq \epsilon$—converges to the finite limit $A(G^*)$ for $\epsilon \longrightarrow 0$. It follows by (4.14) that

$$A_\epsilon(G^{(m)*}) = \frac{1}{4\pi}\int\int_{T_\epsilon} I_m^*.\rho d\theta.\rho d\varphi \tag{4.16}$$

converges *uniformly* to $A(G^{(m)*})$ for $\epsilon \longrightarrow 0$.

Now since I_m^* stays uniformly bounded on T_ϵ in θ, φ and m, in fact

$$I_m^* \leqq \frac{4nR^2}{4\rho^2 \sin^2 \frac{\epsilon}{2}},$$

and furthermore

$$\lim_{m \to \infty} I_m^* = I^*,$$

we have

$$\lim_{m \to \infty} A_\epsilon(G^{(m)*}) = A_\epsilon(G^*);$$

hence

$$\lim_{\epsilon \to 0} \lim_{m \to \infty} A_\epsilon(G^{(m)*}) = \lim_{\epsilon \to 0} A_\epsilon(G^*). \tag{4.17}$$

On account of the uniformity of approach of $A_\epsilon(G^{(m)*})$ to $A(G^{(m)*})$ when $\epsilon \longrightarrow 0$, it is permissible by a well-known mode of reasoning to interchange the order of the limiting processes on the left of (4.17), and we find

$$\lim_{m \to \infty} A(G^{(m)*}) = A(G^*). \tag{4.18}$$

By combining the relations (4.11), (4.13) and (4.18), we see readily that for sufficiently large values of m

$$A(G^{(m)*}) < A(G^{(m)}); \tag{4.19}$$

but this contradicts the fact, proved in §3, that $A(G^{(m)})$ is the minimum value of the A-functional for $\Gamma_\rho^{(m)}$, this being the same as the minimum area bounded by $\Gamma_\rho^{(m)}$.[3]

From the fact, now proved, that $A(G)$ is the minimum value of the A-functional for the contour Γ_ρ, it follows by the relations between the A-functional and area[3] that *the area of* $M_\rho \leqq$ *that of any other continuous surface bounded by* Γ_ρ.

5. Let now Γ' denote an arbitrary continuous closed curve in the interior of M, the image of any Jordan curve C' in the interior of the circular disc $|w| < 1$. On account of the fact that the singular points of M, where

$$\sum_{i=1}^{n} | F_i'(w) |^2 = 0,$$

have no point of condensation in the interior of $|w| < 1$, we can construct with the origin as center a circle C_ρ of radius $\rho < 1$ having on its circumference no singular points and containing C' in its interior.

Denote by M' the region of M_ρ interior to Γ'; if M' is not the surface of least area bounded by Γ', let M'^* be this surface, whose existence is assured by the writer's earlier paper. Then by replacing M' by M'^* we would have a surface bounded by Γ_ρ with an area less than that of M_ρ, which is contrary to the result stated at the end of §4.

[1] "Solution of the Problem of Plateau," *Trans. Amer. Math. Soc.*, **33**, No. 1 (Jan., 1931), pp. 263–321.

[2] Loc. cit., §27.

[3] For the relations between the A-functional and area see the cited paper, §§ 22–26.